

SMALL-X PHYSICS NEAR THE SATURATION REGIME

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We consider the T -matrix near the unitarity limit and the energy dependence of the saturation momentum. We discuss the solution to the Kovchegov equation, or equivalently, to the BFKL evolution in the presence of a single saturation boundary. We include some of the correlations missed in the Kovchegov equation by solving the BFKL equation in the presence of two boundaries. The T -matrix now turns out to be frame-independent, which was not the case for the solution in the case of a single boundary, and it doesn't show the scaling behavior of the solution to the Kovchegov equation. We find for the saturation momentum an energy dependence which differs from the one following from the Kovchegov equation.

1 Introduction

The growth of cross sections with increasing energy, or parton densities with decreasing Bjorken- x , at a fixed hard scale is given by the BFKL¹ evolution when parton occupation numbers are not too large or the T -matrix not too close to the unitarity limit. At or near the unitarity limit nonlinear parton evolution becomes important which is not described by the linear BFKL equation. An extension of the BFKL equation that includes also nonlinear evolution is the Balitsky equation² or the Jalilian-Marian, Iancu, McLerran, Leonidov and Kovner (JIMWLK)³ equation. The Balitsky and JIMWLK equations are coupled equations involving higher and higher correlations and as such are very difficult to deal with analytically. A better understanding of the Balitsky and JIMWLK evolution may emerge from numerical calculations such as⁴.

Kovchegov⁵ has suggested a simpler equation than the Balitsky or JIMWLK equations to deal with scattering at or near the unitarity limit. The Kovchegov equation can be viewed as a mean field version of the Balitsky or JIMWLK equation in which higher correlations are neglected. While incomplete, as is any mean field like approximation, the Kovchegov equation is likely the best equation one can write down in terms of functions which has built in correct unitarity limits for high energy scattering. Many interesting limits of the Kovchegov equation have been understood by analytical methods. In this work we focus on the transition region from

weak to the saturation regime in which Munier and Peschanski⁷ have determined the form of the T -matrix and the rapidity dependence of the saturation momentum by solving the Kovchegov equation analytically. The same results have been obtained even before by the authors of Ref.⁶ by solving the BFKL equation in the presence of a single saturation boundary which effectively approximates the unitarity limit guaranteed in the Kovchegov equation. In the next Section we derive the Kovchegov equation and study its limitations due to the missed higher correlations.

We attempt to go beyond the mean field like approximation in the Kovchegov equation by solving the BFKL equation in the presence of two boundaries¹¹. We find in the vicinity of the saturation regime an expression for the T -matrix which is frame-independent and doesn't show a scaling behavior in contrast to the solution to the Kovchegov equation. The energy dependence of the saturation momentum is also different from the one coming from the Kovchegov equation.

2 The Kovchegov equation

Consider the high-energy scattering of a color dipole on a target (another dipole, hadron, or nucleus) at relative rapidity Y in a frame where the dipole is an elementary quark-antiquark pair and the target a highly evolved system. Now we wish to know how the elastic scattering amplitude $S(\underline{x}_0, \underline{x}_1, Y)$ changes when the rapidity Y is increased by a small amount dY (\underline{x}_0 and \underline{x}_1 are the transverse coordinates of the quark and antiquark of the dipole). The change of $S(\underline{x}_0, \underline{x}_1, Y)$ with rapidity is determined by the Balitsky or JIMWLK equation

$$\frac{\partial}{\partial Y} S(\underline{x}_{01}, Y) = \frac{\alpha N_c}{2\pi^2} \int d^2 \underline{x}_2 \frac{\underline{x}_{01}^2}{\underline{x}_{02}^2 \underline{x}_{12}^2} \left[S^{(2)}(\underline{x}_{02}, \underline{x}_{12}, Y) - S(\underline{x}_{01}, Y) \right]. \quad (1)$$

This equation can be interpreted as follows: When the increase dY is viewed as increasing the rapidity of the dipole then the probability for the dipole to emit a gluon (transverse coordinate \underline{x}_2) increases. In the large N_c limit this quark-antiquark-gluon state can be viewed as a system of two dipoles. The scattering of this two dipole state on the target is given by $S^{(2)}(\underline{x}_{02}, \underline{x}_{12}, Y)$ while the subtracted $S(\underline{x}_{01}, Y)$ is the virtual contribution necessary to normalize the wavefunction¹⁰. The later gives the scattering of a single dipole on the target because the gluon is not in the wavefunction of the dipole at the time of the scattering. The weight in Eq. 1 gives the probability for the emission of two dipoles of sizes \underline{x}_{02} and \underline{x}_{12} from the initial dipole of size \underline{x}_{01} .

It is difficult to use the Balitsky-JIMWLK equation because of the unknown $S^{(2)}(\underline{x}_{02}, \underline{x}_{12}, Y)$. The assumption that the scattering of the two dipole state on the target factorizes

$$S^{(2)}(\underline{x}_{02}, \underline{x}_{12}, Y) = S(\underline{x}_{02}, Y) S(\underline{x}_{12}, Y), \quad (2)$$

which is a sort of a mean field approximation for the gluonic fields in the target, leads to the Kovchegov equation⁵

$$\frac{\partial}{\partial Y} S(\underline{x}_{01}, Y) = \frac{\alpha N_c}{2\pi^2} \int d^2 \underline{x}_2 \frac{\underline{x}_{01}^2}{\underline{x}_{02}^2 \underline{x}_{12}^2} [S(\underline{x}_{02}, Y) S(\underline{x}_{12}, Y) - S(\underline{x}_{01}, Y)]. \quad (3)$$

One exact result of the Kovchegov equation for the S -matrix deep in the saturation region is the Levin-Tuchin formula⁸

$$S(\rho, b, Y) \sim e^{-c(\rho - \rho_s)^2} \quad (4)$$

with the constant $c = -C_F(1 - \lambda_0)/(N_c 2\chi(\lambda_0))$. Recently, the authors of Ref.⁹ have claimed that S deep in the saturation regime has the form given by Eq. 4 but with a constant at least a factor of 2 smaller than the c which follows from the Kovchegov equation. The cause for this discrepancy is the lack of fluctuations in the Kovchegov equation.

Now let us focus on the region close to the saturation regime. Consider the scattering of a dipole of size x on a dipole of size x' at relative rapidity Y and impact parameter b . Near

the saturation boundary the Kovchegov equation or the BFKL equation in the presence of an absorbtive saturation boundary give for the T -matrix in laboratory frame

$$T(x, x', Y; b) = T_0(b, x) \left[Q_s^2(Y) x'^2 \right]^{1-\lambda_0} \ln \frac{1}{Q_s^2(Y) x'^2} \exp \left[-\frac{\pi \ln^2(1/Q_s^2(Y) x'^2)}{4\alpha N_c \chi''(\lambda_0) Y} \right] \quad (5)$$

and for the rapidity dependence of the saturation momentum

$$Q_s^2(x, Y; b) = Q_0^2(x, b) \frac{1}{x^2} \frac{\exp \left[\frac{2\alpha N_c}{\pi} \frac{\chi(\lambda_0)}{1-\lambda_0} Y \right]}{[\alpha Y]^{\frac{3}{2(1-\lambda_0)}}} \quad (6)$$

where $\lambda_0 = 0.372$. Near the saturation boundary, i.e., for the size of the probe x'^2 close to $1/Q_s^2(Y)$, the scattering amplitude in (5) shows a scaling behavior since it depends on $Q_s^2(Y) x'^2$; lines with constant $Q_s^2(Y) x'^2$ are lines of constant scattering amplitude.

Recently¹¹ we have shown that fluctuations are important in evolution also in the region near the saturation boundary (“scaling region”). We have found that the completeness relation

$$n(x, x', Y) = \int \frac{d^2 r_2}{2\pi r_2^2} n(x, r_2, Y/2) n(r_2, x', Y/2) , \quad (7)$$

is not satisfied for the dipole number density n which is obtained by using the BFKL evolution in the presence of a single saturation boundary, as in the case of the T -matrix in Eq. 7. This mismatch is equivalent to the statement that the result for the T -matrix in different frames is different. The reason for this mismatch is the mean field like treatment of fluctuations in the Kovchegov equation: On the left hand side of Eq. 7 the evolution from the initial point $(x, y = 0)$ to the final point $(x', y = Y)$ occurs in one step while on the right hand side of Eq. 7 the evolution can be viewed as proceeding in two steps, from $(x, y = 0)$ to $(r_2, Y/2)$ to (x', Y) , and at the intermediate stage of evolution the fluctuations are not properly taken into account.

In terms of BFKL evolution in the presence of a saturation boundary the reason why the completeness relation is not satisfied goes as follows: on the lhs of Eq. 7 the entire evolution occurs in the presence of a fixed boundary Q_s determined by x and y (Fig. 1a) while on the rhs of Eq. 7 the second part of evolution occurs in the presence of a different boundary \hat{Q}_s determined by r_2 and y (Fig. 1b). Since the dipole number density is obtained in the mean field approximation, one expects that the completeness relation should be fulfilled in an approximative way. Indeed, if one requires for the evolution on both sides of Eq. 7 to happen in the presence the same saturation boundary (Fig. 1a), i.e., if one extends the diffusion region for the second part of the evolution on the rhs of Eq. 7 from \hat{Q}_s to Q_s , then the lhs and rhs of Eq. 7 give the same result. However, because of the extended diffusion region, there are now paths of evolution from $(r_2, Y/2)$ to (x', Y) which manifestly violate unitarity ($T \gg 1$). See for more details¹¹.

3 BFKL evolution in the presence of two saturation boundaries

Now we introduce a second absorptive barrier at $\rho_2(y)$, as shown in Fig. 1c, such that, all unitarity violating evolution between the initial point $(x, 0)$ and the final point (x', Y) is eliminated when the evolution is viewed as proceeding in two steps¹¹. The boundary $\rho_1(y)$ corresponds to saturation in the wavefunction of the evolved dipole x scattering on an elementary dipole x' (forward evolution) and $\rho_2(y)$ corresponds to saturation in the wavefunction of the evolved dipole x' scattering on the elementary dipole x (backward evolution). The introduction of the second boundary $\rho_2(y)$ and the symmetry it brings with has another benefit: The T -matrix is now frame-independent which was not the case for the single boundary case.

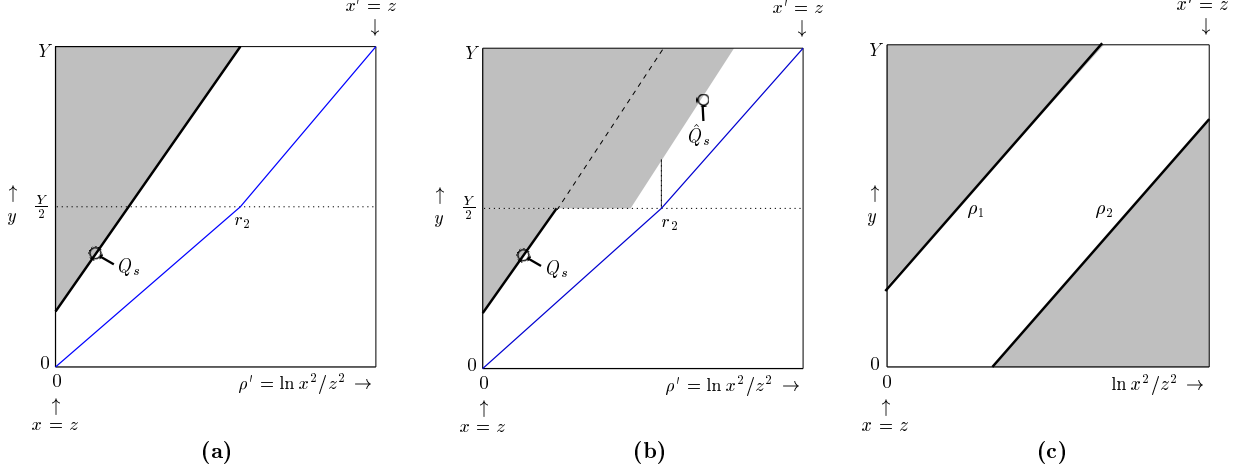


Figure 1: BFKL evolution in the presence of shaded saturation regions in the $Y - \ln(x^2/z^2)$ plane with z generic: (a) evolution in the presence of a single saturation boundary, (b) the second evolution step from $Y/2 \rightarrow Y$ has its own saturation boundary, (c) evolution in the presence of two boundaries.

Near the saturation regime the BFKL evolution in the presence of two absorptive boundaries¹¹ gives a T -matrix which depends on $\ln(Q_s(Y) x')/[\alpha Y/(\Delta\rho)^3]$ with $\Delta\rho \approx \frac{2}{1-\lambda_0} \ln(1/\alpha)$. Thus, the scaling behavior of the solution to the Kovchegov equation is now lost. The rapidity dependence of the saturation momentum for $\alpha Y \gg (\rho_2 - \rho_1)^2/(\pi N_c \chi''(\lambda_d))$ becomes

$$Q_s^2(x, Y; b) = Q_0^2(x, b) \frac{1}{x^2} \exp \left[\frac{2\alpha N_c}{\pi} \frac{\chi(\hat{\lambda}_d)}{1 - \hat{\lambda}_d} \left(1 - \frac{\pi^2}{2\Delta\rho^2} \frac{\chi''(\hat{\lambda}_d)}{\chi(\hat{\lambda}_d)} \left(1 + \frac{c_s}{\Delta\rho} \right) Y \right) \right] \quad (8)$$

with $\hat{\lambda}_d = \lambda_0 + \pi^2/[2(\Delta\rho)^2(1 - \lambda_0)]$ and is different as compared to the result in Eq. 6.

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